

QUANTIZATION AND COHERENT STATES OVER LAGRANGIAN SUBMANIFOLDS

MIKHAIL V. KARASEV

ABSTRACT. A membrane technique, in which the symplectic and Ricci forms are integrated over surfaces in a complexification of the phase space, as well a “creation” connection with zero curvature over lagrangian submanifolds, is used to obtain a unified quantization including a noncommutative algebra of functions, its representations, the Dirac axioms, coherent integral transformations for solutions of spectral and Cauchy problems, and trace formulas.

The results obtained in [1] for the phase space \mathbb{R}^{2n} and Gaussian coherent states (see also [2–5]) are developed in this work for curved phase spaces and general coherent states.

§1. Complexification. Let \mathfrak{X} be a manifold endowed with a complex structure J . By (x, y) we denote points from the product $\mathfrak{X}^\# = \mathfrak{X} \times \mathfrak{X}$, and we equip this space with the complex structure $J \oplus J$ and with the groupoid multiplication $(x, y) \circ (y, w) = (x, w)$. The set of units (diagonal) in $\mathfrak{X}^\#$ is identified with $\mathfrak{X} \approx \text{diag}$; the left and the right reductions $\mathfrak{X} \xleftarrow{\pi} \mathfrak{X}^\# \xrightarrow{\bar{\pi}} \mathfrak{X}$ in this groupoid are given by the projections $\pi(x, y) = x$, $\bar{\pi}(y, x) = x$. These mappings are, respectively, a morphism and an antimorphism of complex structures, and so $\mathfrak{X}^\#$ is a *complex groupoid* corresponding to the manifold (\mathfrak{X}, J) , see [6, 7]. Let $\Pi(x), \bar{\Pi}(x) \subset \mathfrak{X}^\#$ be fibers of π and $\bar{\pi}$ over $x \in \mathfrak{X}$. Following physicists, we denote $(x, y) \equiv y|x$.

Note that one can identify the complexified tangent spaces ${}^{\mathbb{C}}T_x\mathfrak{X}$ with the tangent spaces $T_{x|x}\mathfrak{X}^\#$ by using the natural identifications $T_x\mathfrak{X} \approx T_{x|x}\text{diag}$ and $i \cdot T_x\mathfrak{X} \approx (J \oplus J)T_{x|x}\text{diag}$. Then the eigenspaces of J_x corresponding to the eigenvalues $+i$ and $-i$ coincide with the tangent subspaces $T_{x|x}\Pi(x), T_{x|x}\bar{\Pi}(x)$ in $T_{x|x}\mathfrak{X}^\#$. Thus, the groupoid $\mathfrak{X}^\#$ can be regarded as a *complexification* of the manifold (\mathfrak{X}, J) ; the fibers $\Pi(x)$ can be interpreted as the integral leaves of the complex polarization Π on \mathfrak{X} corresponding to the complex structure J , and the fibers $\bar{\Pi}(x)$ can be interpreted as the leaves of the conjugate polarization $\bar{\Pi}$. The groupoid inversion mapping $x|y \rightarrow y|x$ can be interpreted as an involution in the complexification.

We use $z = \{z^k\}$ to denote local holomorphic coordinates on \mathfrak{X} , and by ∂ we denote the holomorphic differential, $\partial = \partial/\partial z$. Thus, Π is generated by the vectors $\partial/\partial \bar{z}^k$.

§2. Membrane amplitudes. Throughout the sequel \mathfrak{X} is assumed connected and simply connected. We call \mathfrak{X} a *symplectic-Kähler manifold* if it is equipped with a symplectic form ω compatible with the complex structure J , and is also endowed

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$ -TEX

with a Kählerian metric g . Let us choose a lagrangian submanifold $\Lambda \subset \mathfrak{X}$ and suppose that

$$(1) \quad \frac{1}{2\pi} \int_{\Sigma} \left(\frac{\omega}{\hbar} + \rho \right) - \frac{1}{2\pi} \oint_{\partial\Sigma} \nu_{\Lambda} \in \mathbb{Z}, \quad \forall \Sigma \subset \mathfrak{X}, \partial\Sigma \subset \Lambda,$$

where $\hbar > 0$, $\rho = (i/2)\bar{\partial}\partial \ln \det g$ is one half the Ricci form on \mathfrak{X} , and ν_{Λ} is a *fundamental* 1-form on Λ such that $d\nu_{\Lambda} = \rho|_{\Lambda}$; see the details in [7–9]. By Σ we denote *membranes* in \mathfrak{X} , i.e., immersed oriented two-dimensional surfaces with boundary.

We shall also be interested in several kinds of membranes in the complexification $\mathfrak{X}^{\#}$.

Triangle. Connect points $\alpha, \beta \in \Lambda$, and $\beta|\alpha \in \mathfrak{X}^{\#}$ by paths $\alpha \xleftarrow{\Gamma} \beta \xleftarrow{\Gamma} \beta|\alpha \xleftarrow{\Gamma} \alpha$ going along $\Pi(\alpha)$, $\bar{\Pi}(\beta)$, and Λ successively. The last path along Λ will be denoted by $\Gamma = \Gamma(\beta|\alpha)$, and *triangle* $_{\Gamma}(\beta|\alpha)$ will denote a membrane in $\mathfrak{X}^{\#}$ with this circle boundary.

Quadrangle. Connect points $x \xleftarrow{\Gamma} x|y \xleftarrow{\Gamma} y \xleftarrow{\Gamma} y|x \xleftarrow{\Gamma} x$ along $\Pi(x)$, $\bar{\Pi}(y)$, $\Pi(y)$, and $\bar{\Pi}(x)$ successively, and denote by *quadrangle* (y, x) a membrane in $\mathfrak{X}^{\#}$ with this boundary.

Pentagon. Connect points $\alpha \xleftarrow{\Gamma} \beta \xleftarrow{\Gamma} \beta|x \xleftarrow{\Gamma} x \xleftarrow{\Gamma} x|\alpha \xleftarrow{\Gamma} \alpha$ along $\Pi(\alpha)$, $\bar{\Pi}(x)$, $\Pi(x)$, $\bar{\Pi}(\beta)$, and Λ . A membrane with this boundary is denoted by *pentagon* $_{\Gamma}(\beta|x|\alpha)$.

Hexagon. Paths $x \xleftarrow{\Gamma} x|w \xleftarrow{\Gamma} w \xleftarrow{\Gamma} w|y \xleftarrow{\Gamma} y \xleftarrow{\Gamma} y|x \xleftarrow{\Gamma} x$ along $\Pi(x)$, $\bar{\Pi}(y)$, $\Pi(y)$, $\bar{\Pi}(w)$, $\Pi(w)$, $\bar{\Pi}(x)$ generate a membrane *hexagon* (w, y, x) .

To each of the above membranes we relate a *membrane amplitude* as follows:

$$\begin{aligned} a_{\Lambda}(\beta|\alpha) &= \exp \left\{ i \int_{\text{triangle}_{\Gamma}(\beta|\alpha)} \left(\frac{\omega}{\hbar} + \rho \right) - i \int_{\Gamma(\beta|\alpha)} \nu_{\Lambda} \right\} \cdot \Delta_{\Lambda}(\alpha)^{1/2} \Delta_{\Lambda}(\beta)^{1/2}, \\ p(y, x) &= \exp \left\{ i \int_{\text{quadrangle}(y, x)} \left(\frac{\omega}{\hbar} + \rho \right) \right\}, \\ k_{\Lambda}(\beta|x|\alpha) &= \exp \left\{ i \int_{\text{pentagon}_{\Gamma}(\beta|x|\alpha)} \left(\frac{\omega}{\hbar} + \rho \right) - i \int_{\Gamma(\beta|\alpha)} \nu_{\Lambda} \right\} \cdot \Delta_{\Lambda}(\alpha)^{1/2} \Delta_{\Lambda}(\beta)^{1/2}, \\ b(w, y, x) &= \exp \left\{ i \int_{\text{hexagon}(w, y, x)} \left(\frac{\omega}{\hbar} + \rho \right) \right\}. \end{aligned}$$

Here Δ_{Λ} is a *modular function* on Λ [8], and the form $\omega/\hbar + \rho$ is assumed continued analytically on $\mathfrak{X}^{\#}$, possibly with singularities, so that all amplitudes a_{Λ} , p , k_{Λ} , and b are globally defined and smooth. In view of (1), these amplitudes do not depend on the choice of membranes. Note that (1) implies the condition $1/2\pi[\omega/\hbar + \rho] \in H^2(\mathfrak{X}, \mathbb{Z})$, well known in the Kostant-Souriau quantization. We call a_{Λ} a *transition amplitude* and k_{Λ} a *coherent amplitude* over Λ ; we also call p and b a *probability* and a *holonomy* function over \mathfrak{X} [8, 9].

§3. Geometric star product and the completeness axiom. The probability and holonomy functions determine operators \mathcal{P} and \mathcal{B} by the formulas

$$\begin{aligned} \mathcal{P}f(x) &\stackrel{\text{def}}{=} \int_{\mathfrak{X}} p(x, y) f(y) dm(y), \\ \mathcal{B}(f \otimes g) &\stackrel{\text{def}}{=} \int_{\mathfrak{X}} \int_{\mathfrak{X}} b(w, y, x) f(y) g(x) dm(y) dm(x), \end{aligned}$$

where dm is a measure on \mathfrak{X} . The kernel $\text{Ker } \mathcal{P} \subset L^2(\mathfrak{X}, dm)$ is nontrivial in general but the range of \mathcal{B} is orthogonal to $\text{Ker } \mathcal{P}$, so that the element

$$(2) \quad f * g \stackrel{\text{def}}{=} \mathcal{P}^{-1} \mathcal{B}(f \otimes g), \quad \forall f, g \in C_0^\infty(\mathfrak{X})$$

is well defined in the orthogonal complement $(\text{Ker } \mathcal{P})^\perp$. The multiplication $*$ defined by (2) is noncommutative and associative. The completion $\mathcal{L}_\mathfrak{X}$ of the space $(\text{Ker } \mathcal{P})^\perp$ equipped with this multiplication and with the inner product $(f, g)_{\mathcal{L}_\mathfrak{X}} \stackrel{\text{def}}{=} (\mathcal{P}f, g)_{L^2(\mathfrak{X}, dm)}$ is a Hilbert algebra; see details in [6, 10]. This quantum algebra of functions on \mathfrak{X} has the unit 1, i.e., $1 * f = f * 1 = f$, if the probability function p and the measure dm satisfy

$$(3) \quad \int_{\mathfrak{X}} p(x, y) dm(y) = 1, \quad \forall x \in \mathfrak{X}.$$

This is a *completeness* axiom. Certain classes of homogeneous Kähler manifolds automatically satisfy this axiom with respect to the Liouville measure. In general the existence of a measure on \mathfrak{X} with respect to which property (3) would be held is an open question.

In what follows we suppose that (3) holds.

There is a simple formula for the multiplication (2) in terms of local complex coordinates. First, note that one can express the measure dm via local coordinates, $dm = M d\bar{z} dz$, and define the *dual measure* $dm' \stackrel{\text{def}}{=} M' d\bar{z} dz$ using the density $M' = M^2 / \det g$. If there is a Kählerian metric g' on \mathfrak{X} such that $M' = \det g'$, then g' is referred to as a *dual metric*.

One can represent the symplectic form as $\omega = i g_{k\bar{s}}^\omega d\bar{z}^k \wedge dz^s$, where $g_{k\bar{s}}^\omega = \bar{\partial}_k \partial_s F$; here F is a local potential. The corresponding Liouville measure and “symplectic” Laplace operator on \mathfrak{X} are denoted by $dm^\omega = \det g^\omega \cdot d\bar{z} dz$ and ${}^\omega \Delta = 2(g^\omega)^{-1} \bar{\partial} \partial$. Let us take a function $H \in C^\infty(\mathfrak{X})$ and define its *local holomorphic realization* $\mathcal{H} = \mathcal{H}(\xi, z)$ by setting $H = M'^{-1/2} \circ \mathcal{H}(\partial F - \hbar \partial, z)(M'^{1/2})$, where $\partial \equiv \partial / \partial z$. If $\mathcal{H}(\xi, z)$ is a polynomial in the momenta ξ , then $\mathcal{H}(\partial F - \hbar \partial, z)$ is well defined as a Weyl-symmetrized operator.

Theorem 1. *For functions H whose local holomorphic realizations \mathcal{H} are polynomial in the momenta, the operator of left multiplication in the algebra (2) is given by the formula*

$$H * = M'^{-1/2} \circ \mathcal{H}(\partial F - \hbar \partial, z) \circ M'^{1/2} = H - i\hbar \text{ad}(H)_+ + \hbar^2 D(H)_{++}.$$

Here $\text{ad}(H)_+$ is the ∂ -part of the Hamilton field $\text{ad}(H)$, and $D(H)_{++}$ is an operator regularly depending on \hbar and annihilating antiholomorphic functions.

§4. Evaluation of the star product. Holomorphic filtration. Let $\text{ad}(H)$ denote the Hamilton field related to H and ω . Let $\mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$ be the subspace of functions whose Hamilton flows preserve the polarization, i.e., $H \in \mathcal{F}^{(1)} \Leftrightarrow [\text{ad}(H), \Pi] \subset \Pi$. By analogy, we define the space $\mathcal{F}^{(2)}$ as follows: $H \in \mathcal{F}^{(2)} \Leftrightarrow [[\text{ad}(H), \Pi], \Pi] \subset \Pi$, and so on. The sequence $\mathcal{F}^{(1)} \subset \mathcal{F}^{(2)} \subset \dots$ will be called a *holomorphic filtration*. The first space $\mathcal{F}^{(1)}$ is a Lie algebra with respect to the Poisson brackets.

Let us (locally) define a 2-tensor $((\mathbb{H}^{sk}))$ by the relation $\{\{H, f\}, g\} = \mathbb{H}^{sk} \partial_s f \partial_k g$ for any holomorphic functions f and g . This tensor is the $\bar{\partial}\partial$ -part of the first variation of $ad(H)$. Note that $H \in \mathcal{F}^{(1)} \Leftrightarrow \mathbb{H} \equiv 0$, and $H \in \mathcal{F}^{(2)} \Leftrightarrow \mathbb{H}$ is holomorphic. We consider three special cases:

(I) $H \in \mathcal{F}^{(1)}$; (II) $H \in \mathcal{F}^{(2)}$, and the dual metric g' on \mathfrak{X} is Ricci-flat; (III) $H \in \mathcal{F}^{(2)}$ and \mathbb{H} is nondegenerate, i.e., all local tensors $((\mathbb{H}^{ks}))$ are invertible.

In cases (I) and (II), let us define a *quantum deformation* as follows [7]:

$$H_{\hbar} \stackrel{\text{def}}{=} H - \hbar^\omega \Delta(H)/4 - i\hbar ad(H)_+ (\ln |\mathcal{D}m'/\mathcal{D}m^\omega|/2).$$

In case (III) the inverse \mathbb{H}^{-1} transforms under changes of coordinates z like a metric. Thus, there is the ‘‘Laplace operator’’ ${}^{\mathbb{H}}\Delta$ with respect to this ‘‘metric’’. Let us denote

$$(4) \quad \mathbb{D}(H) = \frac{1}{2}(\det \mathbb{H})^{-1/4} \circ {}^{\mathbb{H}}\Delta \circ (\det \mathbb{H})^{1/4}, \quad {}^{\mathbb{H}}\Delta \stackrel{\text{def}}{=} (\det \mathbb{H})^{1/2} \partial_s \circ (\det \mathbb{H})^{-1/2} \mathbb{H}^{sk} \circ \partial_k,$$

and define a secondary quantum deformation $H_{\hbar\hbar} \stackrel{\text{def}}{=} H_{\hbar} + \hbar^2 (M')^{-1/2} \mathbb{D}(H) (M'^{1/2})$.

Theorem 2. (I) If $H \in \mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$, then $H_{\hbar}^* = H_{\hbar} - i\hbar ad(H)_+$.

(II) If $H \in \mathcal{F}^{(2)}(\mathfrak{X}, \Pi)$ and the dual metric g' on \mathfrak{X} is Ricci-flat, then $H_{\hbar}^* = H_{\hbar} - i\hbar ad(H)_+ + \hbar^2 (\det g')^{-1} \partial_s \circ (\det g') \mathbb{H}^{sk} \circ \partial_k$.

(III) If $H \in \mathcal{F}^{(2)}(\mathfrak{X}, \Pi)$ and the $\bar{\partial}\partial$ -variation of the field $ad(H)$ is nondegenerate, then $H_{\hbar\hbar}^* = H_{\hbar} - i\hbar ad(H)_+ + \hbar^2 (M')^{-1/2} \circ \mathbb{D}(H) \circ (M')^{1/2}$.

The statement (I), as well as the following Corollary, was obtained in [7].

Corollary 1. The quantum deformation is a homomorphism of the Lie algebra $\mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$ into the associative algebra (2): $H_{\hbar} * G_{\hbar} - G_{\hbar} * H_{\hbar} = -i\hbar \{H, G\}_{\hbar}$, $\forall H, G \in \mathcal{F}^{(1)}$.

§5. Geometric inner product and quantization over lagrangian submanifolds. The transition and coherent amplitudes determine integral operators \mathcal{A}_{Λ} and \mathcal{K}_{Λ} over Λ :

$$\mathcal{A}_{\Lambda} \varphi(\beta) \stackrel{\text{def}}{=} \int_{\Lambda} a_{\Lambda}(\beta|\alpha) \varphi(\alpha) d\sigma(\alpha), \quad (\mathcal{K}_{\Lambda}(x)\varphi)(\beta) \stackrel{\text{def}}{=} \int_{\Lambda} k_{\Lambda}(\beta|x|\alpha) \varphi(\alpha) d\sigma(\alpha),$$

where $d\sigma$ is a measure on Λ . The kernel $\text{Ker } \mathcal{A}_{\Lambda} \subset L^2(\Lambda, d\sigma)$ is nontrivial in general, but the range of $\mathcal{K}_{\Lambda}(x)$ ($\forall x \in \mathfrak{X}$) is orthogonal to $\text{Ker } \mathcal{A}_{\Lambda}$, so that the operators

$$(5) \quad \mathcal{C}_{\Lambda}(x) \stackrel{\text{def}}{=} \mathcal{A}_{\Lambda}^{-1} \mathcal{K}_{\Lambda}(x)$$

are well defined in the orthogonal complement $(\text{Ker } \mathcal{A}_{\Lambda})^{\perp}$. The completion of $(\text{Ker } \mathcal{A}_{\Lambda})^{\perp}$ with respect to the inner product $(\varphi_1, \varphi_2)_{\Lambda} \stackrel{\text{def}}{=} (\mathcal{A}_{\Lambda} \varphi_1, \varphi_2)_{L^2(\Lambda, d\sigma)}$ will be denoted by \mathcal{L}_{Λ} ; see the details in [8] and [9, 11].

Theorem 3. Let Λ be a closed lagrangian submanifold in a symplectic-Kähler manifold \mathfrak{X} , and let conditions (1) and (3) hold. Then

(i) the mapping $\mathcal{C}_{\Lambda} : \mathfrak{X} \rightarrow \text{Hom } \mathcal{L}_{\Lambda}$ defined in (5) is a quantization of \mathfrak{X} represented in the Hilbert space \mathcal{L}_{Λ} of functions on Λ . This means that [7]:

- $\mathcal{C}_\Lambda(x)$ is a one-dimensional orthogonal projection in \mathcal{L}_Λ for any $x \in \mathfrak{X}$;
- the first quantum correlator generated by \mathcal{C}_Λ coincides with the probability function over \mathfrak{X} , i.e., $\text{tr}(\mathcal{C}_\Lambda(x)\mathcal{C}_\Lambda(y)) = p(x, y)$;
- the family \mathcal{C}_Λ is smooth and complete, i.e., $\int_{\mathfrak{X}} \mathcal{C}_\Lambda(x) dm(x) = I$.
- (ii) The dimensions are given by the formulas $\dim \mathcal{L}_\Lambda = (\dim \mathcal{L}_{\mathfrak{X}})^{1/2} = \int_{\mathfrak{X}} dm$.
- (iii) The correspondence between functions on \mathfrak{X} and operators over Λ , defined by

$$(6) \quad H \rightarrow \overset{\vee}{H}, \quad \overset{\vee}{H} \stackrel{\text{def}}{=} \int_{\mathfrak{X}} H(x) \mathcal{C}_\Lambda(x) dm(x),$$

possesses the following properties: $\overset{\vee}{1} = I$, $\overset{\vee}{H}\overset{\vee}{G} = \overset{\vee}{H} * \overset{\vee}{G}$, $\overset{\vee}{H}^* = \overset{\vee}{\overline{H}}$, where the star product $*$ is defined in (2). Thus, the mapping $H \rightarrow \overset{\vee}{H}$ gives a representation of the quantum algebra $\mathcal{L}_{\mathfrak{X}}$ in the Hilbert space \mathcal{L}_Λ .

(iv) For functions H whose local holomorphic realizations \mathcal{H} are polynomial in the momenta, the operator (6) is given by the formula:

$$(7) \quad \overset{\vee}{H} = q(\alpha)^{1/2} \circ \mathcal{H}(\partial F(\alpha) - \hbar \tilde{\partial}, z(\alpha)) \circ q(\alpha)^{-1/2}.$$

Here $q(\alpha) \stackrel{\text{def}}{=} \mathcal{D}z(\alpha)/\mathcal{D}\sigma(\alpha)$, $\alpha \in \Lambda$, $\{z = z(\alpha)\}$ is the local equation of the submanifold Λ , $\tilde{\partial} \stackrel{\text{def}}{=} (\partial z/\partial \alpha)^{-1*} \partial/\partial \alpha$. The operators $\partial F - \hbar \tilde{\partial}$ and $z(\alpha)$ in (7) are Weyl-symmetrized.

Formulas (6), (7) represent the general construction of quantum operators over a lagrangian submanifold. For functions H of the first or second holomorphic filtration one can evaluate the operator $\overset{\vee}{H}$ explicitly as a first- or second-order differential operator on Λ .

Theorem 4. For $H \in \mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$ the operator $\overset{\vee}{H}_{\hbar}$ on Λ defined by (6) has the form:

$$(8) \quad \overset{\vee}{H}_{\hbar} = H|_{\Lambda} - i\hbar \left(v(H) + \frac{1}{2} \text{div}^{\sigma} v(H) \right).$$

Here $v(H)$ is the projection of $\text{ad}(H)$ on Λ along Π , and div^{σ} denotes the divergence with respect to $d\sigma$. The mapping $H \rightarrow \overset{\vee}{H}_{\hbar}$ satisfies the Dirac axioms:

$$(9) \quad [\overset{\vee}{H}_{\hbar}, \overset{\vee}{G}_{\hbar}] = -i\hbar \{H, G\}_{\hbar}, \quad \overset{\vee}{H}_{\hbar}^* = \overset{\vee}{\overline{H}_{\hbar}}, \quad \forall H, G \in \mathcal{F}^{(1)}(\mathfrak{X}, \Pi).$$

The quantization (8) on $\mathcal{F}^{(1)}$ and the implication (8) \Rightarrow (9) were obtained in [5, 8] independently from the “associative” quantization (6). The general formulas (6) and (7) are valid in more complicated cases. Let us consider, for instance, case (III) listed in §4.

Theorem 5. Suppose that $H \in \mathcal{F}^{(2)}(\mathfrak{X}, \Pi)$ and the $\overline{\partial}\partial$ -variation of the Hamilton field $\text{ad}(H)$ is nondegenerate. Then the operator $\overset{\vee}{H}_{\hbar\hbar}$ on Λ defined by (6) has the form

$$(10) \quad \overset{\vee}{H}_{\hbar\hbar} = H|_{\Lambda} - i\hbar \left(v(H) + \frac{1}{2} \text{div}^{\sigma} v(H) \right) + \frac{\hbar^2}{2} q^{1/2} \circ \mathbb{D}(H) \circ q^{-1/2}.$$

Here the operator $\widetilde{\mathbb{D}}(H)$ is defined by Eqs. (4) with \mathbb{H} replaced by $\widetilde{\mathbb{H}} = \mathbb{H}|_{\Lambda}$ and ∂ by $\widetilde{\partial}$ (where $\widetilde{\partial}$ and q are introduced in Theorem 3 (iv)).

The last summand in (10) is a second-order operator on $\Lambda \subset \mathfrak{X}$. Resembling operators, but defined only in local charts on $\Lambda \subset \mathbb{R}^{2n}$, were met in Maslov's theory of semiclassical approximations. Operators in a sense similar to (10) are also appeared in the Blattner–Kostant–Sternberg and Atiyah–Hitchin approaches to geometric quantization. It is remarkable that by using operators of type (10) one can describe irreducible representations of certain physically important algebras with quadratic commutation relations [12].

§6. Connection with zero curvature over lagrangian submanifolds. Let $\mathbf{P} : \mathfrak{X} \rightarrow \text{Hom } L$ be an abstract quantization of the symplectic–Kähler manifold \mathfrak{X} represented in a Hilbert space L (i.e., the properties mentioned in Theorem 3 (i) hold; see also [7]). Then we obtain a correspondence between functions on \mathfrak{X} and operators, in L ,

$$(11) \quad H \rightarrow \widehat{H}, \quad \widehat{H} \stackrel{\text{def}}{=} \int_{\mathfrak{X}} H(x) \mathbf{P}(x) dm(x)$$

with the following properties: $\widehat{1} = I$, $\widehat{H}\widehat{G} = \widehat{H * G}$, $\widehat{H}^* = \widehat{\overline{H}}$, where the star product $*$ is defined in (2). In Berezin's terminology, the function H is called the contravariant symbol of the operator \widehat{H} (see references and discussion in [6, 7, 10]).

Let us consider the following 1-forms on \mathfrak{X} with values in the space of functions over \mathfrak{X} : $a_x^+(\cdot) \stackrel{\text{def}}{=} \hbar \partial_x \ln p(x, \cdot)$, $a_x^-(\cdot) \stackrel{\text{def}}{=} \hbar \overline{\partial}_x \ln p(x, \cdot)$, where $x \in \mathfrak{X}$ and p is the probability function. By applying to a^+ and a^- the quantization mapping (11), one obtains 1-forms \widehat{a}^+ and \widehat{a}^- on \mathfrak{X} with values in the space of operators acting in L . In view of the identities $\hbar \partial \mathbf{P} = \widehat{a}^+ \mathbf{P}$, $\widehat{a}^- \mathbf{P} = 0$, we call \widehat{a}^+ and \widehat{a}^- the *creation* and *annihilation* forms on \mathfrak{X} (see in [7]).

Theorem 6. *Suppose that \mathbf{P} is a quantization of the symplectic–Kähler manifold $(\mathfrak{X}, J, \omega, g)$, represented in a Hilbert space L , and let $\Lambda \subset \mathfrak{X}$ be a lagrangian submanifold with property (1). Then the following operator-valued 1-form*

$$(12) \quad \widehat{\theta} \stackrel{\text{def}}{=} -\frac{1}{\hbar} \widehat{a}^+|_{\Lambda} + i\nu_{\Lambda} \cdot I$$

on Λ determines a connection with zero curvature and with identity global holonomy in the trivial L -bundle over Λ (i.e., in $\Lambda \times L$). In formula (12), I is the identity operator in L , and ν_{Λ} is the fundamental 1-form on Λ .

If one fixes a point $\alpha_0 \in \Lambda$ and a “vacuum” vector $e_{\alpha_0} \in L$ such that $\mathbf{P}(\alpha_0) = e_{\alpha_0} \otimes e_{\alpha_0}^*$, then the parallel translation of e_{α_0} along paths $\alpha_0 \rightarrow \alpha$ on Λ by means of the connection (12) gives a smooth family of vectors in L : $e_{\alpha} = \text{Exp}\left\{-\int_{\alpha_0}^{\alpha} \widehat{\theta}\right\} e_{\alpha_0}$, $\alpha \in \Lambda$ with the following properties: $\mathbf{P}(\alpha) = e_{\alpha} \otimes e_{\alpha}^*$, $\widehat{a}_{\alpha}^- e_{\alpha} = 0$. If one defines

$$(13) \quad u_{\alpha} \stackrel{\text{def}}{=} \Delta_{\Lambda}(\alpha)^{1/2} e_{\alpha},$$

then $(u_{\alpha}, u_{\beta})_L = a_{\Lambda}(\beta|\alpha)$, where a_{Λ} is the transition amplitude and Δ_{Λ} is the modular function on Λ .

§7. Coherent integral transformation. Using the family of vectors (13), we determine the integrals

$$(14) \quad U_\Lambda(\varphi) \stackrel{\text{def}}{=} \int_\Lambda \varphi(\alpha) u_\alpha d\sigma(\alpha), \quad \varphi \in C_0^\infty(\Lambda).$$

Theorem 7. *The transformation (14) generates a unitary isomorphism $U_\Lambda : \mathcal{L}_\Lambda \rightarrow L$ intertwining quantizations (6) and (11), i.e., $\hat{H} \circ U_\Lambda = U_\Lambda \circ \check{H}$.*

Following [11], we call vectors $u_\alpha \in L$ *geometric coherent states* over Λ represented in the Hilbert space L (or, in short, Λ -coherent states in L). The mapping U_Λ will be called a *coherent integral transformation* over Λ .

Corollary 2. *If φ is an eigenfunction of the operator \check{H} in \mathcal{L}_Λ , then $U_\Lambda(\varphi)$ is the eigenvector of the operator \hat{H} in L related to the same eigenvalue.*

This statement is the basis for constructing useful formulas as well as for solving equations in mathematical physics. If the phase space $(\mathfrak{X}, J, \omega, g)$, the submanifold $\Lambda \subset \mathfrak{X}$ and the quantization $\mathbf{P} : \mathfrak{X} \rightarrow \text{Hom}(L)$ are taken in an appropriate way, integral (14) leads to many well-known integral representations of special functions.

From the other hand the quantization procedure itself on the level of operators, star products, etc., can be expressed in terms of transformations U_Λ (14) and operators $\check{H} = \check{H}_\Lambda$ (6). For instance, one has the following formulas $\hat{H} = U_{\text{diag}}(H)$, $H* = \check{H}_{\text{diag}}$, where diag denotes the lagrangian diagonal in $\mathfrak{X}^\# = \mathfrak{X} \times \mathfrak{X}$ (note that $\mathfrak{X}^\#$ is, in a natural way, a symplectic-Kähler groupoid); see in [5, 7].

The coherent transformation (14) is also very effective for the theory of semiclassical approximation (what was the basic observation and the starting point in [1]; see also [2–5, 9, 13–17]). In the semiclassical approach the number \hbar is a parameter tending to zero over a certain subset $\mathcal{R} \subset (0, \infty)$, $0 \in [\mathcal{R}]$. In the general case, the metric on \mathfrak{X} can depend on this parameter $g = g_{(\hbar)}$ as a regular series in $\hbar \rightarrow 0$, and ω is supposed to be the Kähler form for the metric $g_{(0)}$. The rule (1) and axiom (3) are assumed for each $\hbar \in \mathcal{R}$. The measures $dm = dm_\hbar$ on \mathfrak{X} and $d\sigma = d\sigma_\hbar$ on Λ must be represented as follows: $dm_\hbar = (2\pi\hbar)^{-\dim \mathfrak{X}/2} (dm^\omega + O(\hbar))$, $d\sigma_\hbar = (4\pi\hbar)^{-\dim \mathfrak{X}/4} (d\sigma^0 + O(\hbar))$.

Theorem 8. *Under the above assumptions the following asymptotic expansions hold as $\hbar \rightarrow 0$. For the star product (2): $f*g = fg - i\hbar \text{ad}(f)_+ g + O(\hbar^2)$, $f*g - g*f = -i\hbar \{f, g\} + O(\hbar^2)$. For the quantum operators (6) over lagrangian submanifolds: $\check{H} = H|_\Lambda - i\hbar(v(H) + \text{div}^{\sigma^0} v(H)/2) + O(\hbar^2)$; and, moreover, if $H|_\Lambda \equiv \lambda_0 = \text{const}$ and $d\sigma^0$ is invariant with respect to the Hamilton flow γ_H^τ , then $\check{H} = \lambda_0 - i\hbar d/d\tau + O(\hbar^2)$. In the latter case the vector $u = U_\Lambda(1)$ is an approximate eigenvector (quasimode) for the operator \hat{H} corresponding to an eigenvalue $\lambda_0 + O(\hbar^2)$, i.e., $\hat{H}u = \lambda_0 u + O(\hbar^2)$, $\|u\|^2 = \int_\Lambda d\sigma^0 + O(\hbar)$.*

§8. Cauchy problem and trace formula. The coherent transformation (14) can be applied not only to the eigenvalue problem, but to the Cauchy problem as well [2, 5, 16, 17]. Let us take a real smooth function H on \mathfrak{X} . The graph $(\gamma_H^t) = \{(\gamma_H^t(x), x) \mid x \in \mathfrak{X}\}$ is a lagrangian submanifold in $\mathfrak{X}^\# = \mathfrak{X} \times \mathfrak{X}$. Let us transport

the dual metric dm' from \mathfrak{X} onto the graph (γ_H^t) by means of the projection $\mathfrak{X}^\# \xrightarrow{\pi} \mathfrak{X}$. The quantum operator (6) related to the graph $(\gamma_H^t) \subset \mathfrak{X}^\#$ and to the function $H^\# = H \otimes 1$ on $\mathfrak{X}^\#$ can be represented as follows: $\overset{\vee}{H}^\# = H - i\hbar ad(H)_+ + \hbar^2 D^t(H)_{++}$. The latter operator $D^t(H)_{++}$ vanishes if $H \in \mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$, and it is a second order differential operator (in x) if $H \in \mathcal{F}^{(2)}(\mathfrak{X}, \Pi)$; see Theorems 4,5. We denote by $f^t(x)$ the solution of the Cauchy problem over the manifold \mathfrak{X} :

$$(15) \quad i\partial f^t / \partial t = \hbar D^t(H)_{++} f^t, \quad f^t|_{t=0} = f.$$

Then the solution of the Cauchy problem in the Hilbert space L is expressed as follows.

Theorem 9. *Let the operator \hat{H} (11) be selfadjoint in the Hilbert space L . Then for any $f \in C_0^\infty(\mathfrak{X})$ one has the formula $\exp\{-it\hat{H}/\hbar\}\hat{f} = U_{\text{graph}(\gamma_H^t)}(f^t) = U_{\text{graph}(\gamma_H^t)}(f) + O(\hbar)$, where the coherent integral transformation related to the graph $(\gamma_H^t) \subset \mathfrak{X} \times \mathfrak{X}$ is defined by (14), and f^t is defined by (15). If $H \in \mathcal{F}^{(1)}(\mathfrak{X}, \Pi)$, then $f^t \equiv f$.*

Corollary 3 (Trace formula).

$$\begin{aligned} \text{tr} \left(e^{-it\hat{H}/\hbar} \hat{f} \right) = \int_{\mathfrak{X}} \exp \left\{ i \int_{\Sigma^t(x)} \left(\frac{\omega}{\hbar} + \rho \right) - i \int_0^t \nu^\tau(x) d\tau \right. \\ \left. - \frac{it}{\hbar} H(x) \right\} \Delta^t(x)^{1/2} f^t(x) dm(x). \end{aligned}$$

Here $\Sigma^t(x)$ is a triangle membrane with the boundary $x \longleftarrow x|\gamma^t(x) \longleftarrow \gamma^t(x) \longleftarrow x$ (see §2, the first path is the trajectory of the Hamilton flow), ν^t is a fundamental form along the trajectory:

$$\nu^t = \frac{i}{4} \frac{\partial}{\partial t} \ln \frac{\mathcal{D}z^t / \mathcal{D}z}{\mathcal{D}\bar{z}^t / \mathcal{D}\bar{z}} - \frac{1}{2} \Im[(\partial \ln \det g)(z^t) \bar{z}^t], \quad z^t \equiv z(\gamma^t(x)), \quad z \equiv z(x),$$

and Δ^t is a modular function along the trajectory:

$$\Delta^t = |\mathcal{D}z^t / \mathcal{D}z| (\det g(\gamma^t(x)) / \det g(x))^{1/2}.$$

In particular, if the flow γ_H^t preserves the Kählerian structure on \mathfrak{X} , then

$$\text{tr} \left(e^{-it\hat{H}/\hbar} \hat{f} \right) = \int_{\mathfrak{X}} \exp \left\{ i \int_{\Sigma^t(x)} \left(\frac{\omega}{\hbar} + \rho \right) - i \int_0^t \nu^\tau(x) d\tau - \frac{it}{\hbar} H(x) \right\} f(x) dm(x).$$

REFERENCES

1. Karasev, M., *Connections over lagrangian submanifolds and certain problems in semiclassical approximation*, Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI) **172** (1989), 41–54; English transl. in *J. Sov. Math.* **59** (1992), 1053–1062.
2. Karasev, M., *To the Maslov theory of quasiclassical asymptotics. Examples of new global quantization formula applications* (1989), Prepr. Inst. Theor. Phys., Kiev, ITP–89–78E, 32 pp..

3. Karasev, M., *New global asymptotics and anomalies for the problem of quantization of the adiabatic invariant*, *Funct. Anal. Appl.* **24** (1990), no. 2, 104–114.
4. Karasev, M., *Quantization by parallel translation. Global formula for semiclassical wave functions*, *Quantum Field Theory, Quantum Mechanics and Optics*, Nova Sci Publ., N. Y., 1991, pp. 189–192.
5. Karasev, M., *Simple quantization formula*, *Symplec. Geom. and Math. Phys.*, Actes du colloque en l'honneur de J.-M. Souriau (P. Donato, C. Duval, J. Elhadad, G. M. Tuynman, eds.), Birkhäuser, Boston, 1991, pp. 234–244.
6. Karasev, M., *Formulas for noncommutative products of functions in terms of membranes and strings. I*, *Russ. J. Math. Phys.* **2** (1994), no. 4, 445–462.
7. Karasev, M., *Geometric coherent states, membranes, and star products*, *Quantiz., Coherent States, and Complex Structures*, Plenum, N. Y., 1995.
8. Karasev, M., *Integrals over membranes, transition, amplitudes, and quantization*, *Russ. J. Math. Phys.* **1** (1993), no. 4, 523–526.
9. Karasev, M., *Quantization by means of two-dimensional surfaces (membranes). Geometrical formulas for wave functions*, *Contemp. Math* **179** (1994), 83–113.
10. Karasev, M., *Geometric star-products*, *Contemp. Math* **179** (1994), 115–121.
11. Karasev, M., *Quantization by membranes and integral representations of wave functions*, *Quantiz. and Inf.-Dimensional Systems*, Plenum, N. Y., 1994, pp. 9–19.
12. Karasev, M. and Novikova, E., *Quadratic Poisson brackets in the Zeeman effect. Irreducible representations and coherent states*, *Uspekhi matem. nauk* **49** (1994), no. 5, 169–170;; Engl. transl. in *Russ. Math. Surveys*.
13. Karasev, M. and Vorobjev, Yu. M., *Hermitian bundles over isotropic submanifolds and correction to Kostant-Souriau quantization rule* (1990), Preprint ITP-90-85E, Inst. Theor. Phys., Kiev.
14. Karasev, M. and Vorobjev, Yu. M., *Integral representations over isotropic submanifold and equations of zero curvature* (1992), Preprint AMath.-QDS-92-01, Moscow Inst. of Electron. & Mathem., Moscow, 56 pp. (to appear in *Adv. Math.*).
15. Karasev, M. and Vorobjev, Yu. M., *Connection and excited wave packets over invariant isotropic torus*, *Quantiz. and Coherent States Methods* (S. Ali, I. Mladenov, A. Odziejewicz, eds.), World Scientific, 1993, pp. 179–189.
16. Karasev, M. and Kozlov, M. B., *Quantum and semiclassical representations over lagrangian submanifolds in $su(2)^*$, $so(4)^*$, $su(1, 1)^*$* , *J. Math. Phys.* **34** (1993), 4986–5006.
17. Karasev, M. and Kozlov, M. B., *Representation of compact semisimple Lie algebras over lagrangian submanifolds*, *Funkt. Anal. i Prilozhen.* **28** (1994), no. 4, 16–27; English translation in *Funct. Anal. and Appl.* **28**, no. 4.

MOSCOW INSTITUTE OF ELECTRONICS AND MATHEMATICS, 3/12 B. VUZOVSKY PER.,
MOSCOW 109028, RUSSIA; E-MAIL: KARASEV@APLMAT.MIEMSTU.MSK.EDU, ROOT@ONTI.MIEM.MSK.SU